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R-matrices and the magic square

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Abstract
The distinguished representations associated with the rows of the Freudenthal magic square have a uniform tensor product graph with edges labelled by linear functions of the dimension of the corresponding division algebra.

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1. Introduction

The idea of a series of Lie algebras goes back to the origins of the representation theory of the classical groups. For example, the Lie algebras $SL(n)$ are regarded as a series in the sense that the irreducible representations can be parametrized so that tensor product decompositions and plethysms can be written uniformly and dimensions of representations are written as rational functions of n.

The suggestion that is the motivation for this work is that the Lie algebras in each row of the Freudenthal magic square should also form a series in this sense. This idea was explored explicitly and systematically in [Cvi84, Cvi77]. These works are currently unobtainable but this work can be found in [Cvi]. Unfortunately this work seems to have gone unnoticed. Independently of this, the exceptional series of Lie groups was studied in [Del96, DdM96, CdM96]. In these papers the exceptional series are regarded as a finite series of simple Lie algebras which include all five of the exceptional simple Lie algebras. The parameter for this series can be taken to be the dual Coxeter number. The main results are several formulae for dimensions of representations written as rational functions in the dual Coxeter number. These dimension formulae were found by computer calculations.

The next development was the paper [LM01]. This paper and [LM02] contain detailed information on each row regarded as a series and are background references for our discussion on each row. The parameter here is the dimension, m, of the corresponding algebra and this is the parameter we will be using in this paper. The main result is to give infinitely many dimension formulae, for all rows except the first. These dimension formulae are deduced from the Weyl dimension formula and do not require any computer calculations.

These series are all one dimensional in the sense that there is a single parameter. The preprint [Vog99] includes, in the same spirit as for the exceptional series, evidence for a two-dimensional series of simple Lie algebras. Although this work was motivated by a problem on Vassiliev invariants of links and has not been published, it has been influential and may have inspired [Del96]. This plane contains the exceptional series as a line and we discuss the exceptional series in this context.

The motivation for this paper is the proposal that each of these series of Lie algebras can be regarded as a series of quantum groups. The initial evidence for this proposal is that the dimension formulae in [LM01] are deduced from the Weyl dimension formula and so this also gives quantum dimension formulae. These are rational functions of two indeterminates q and q^m.

The starting point for this work is the observation that the values of the quadratic Casimir on the composition factors of $V \otimes V$ can be written as linear combinations of m. The eigenvalues of the universal R-matrix (without spectral parameter) of the quantum group acting on $V \otimes V$ are powers of q whose exponents are simple linear combinations of the values of the Casimir. This constructs representations of the braid groups over the field of rational functions in two indeterminates, q and q^m. It is now natural to ask if we can introduce the spectral parameter. This is the question we study in this paper.

The results of this paper support the proposal that these series can be regarded as series of quantum groups. In all cases except the first row of the Freudenthal magic square the spectral parameter can be introduced. In these cases the Yang–Baxter equation is a powerful tool for working with the centralizer algebras of the tensor powers of the distinguished representation V.

In this paper we look at each row of the Freudenthal magic square from the point of view of the R-matrix with spectral parameter. We distinguish two versions of the Yang–Baxter equation. A trigonometric solution is associated with a finite-dimensional representation of a quantum affine algebra and is written as

$$R_1(u)R_2(uv)R_1(v) = R_2(v)R_1(uv)R_2(u)$$

and a rational solution is associated with a finite-dimensional representation of a Yangian and is written as

$$R_1(x)R_2(x+y)R_1(y) = R_2(y)R_1(x+y)R_2(x).$$

Each trigonometric solution gives a rational solution by putting $u = q^x$ and then taking the limit $q \to 1$.

The main method of construction of R-matrices is given by the tensor product graph, see [ZGB91, DGZ94, OW86]. Let V and W be representations of a simple Lie algebra which admit an action of the affine Lie algebra which extends the action of the Lie algebra. This condition implies that there exists an R-matrix for the tensor product. Assume further that the decomposition of the tensor product is multiplicity free. This condition implies that the R-matrix is determined by its eigenvalues. Then the tensor product graph gives a method for determining the eigenvalues. There is a bipartite graph constructed as follows. Each composition factor of $V \otimes W$ is given a parity. In the case $V = W$ this corresponds to the decomposition of $V \otimes V$ into the symmetric square $S^2(V)$ and the exterior square $\Lambda^2(V)$. These are the vertices of the graph. Then two vertices U_1 and U_2 of opposite parity are connected by an edge if U_2 is a composition factor of $\mathfrak{g} \otimes U_1$ (where $\mathfrak{g}$ is the adjoint representation). Then the vertices are labelled by the values of the quadratic Casimir, and each directed edge is labelled by the difference of Casimirs.

If the R-matrix exists then the labels on the edges of the graph satisfy a consistency condition for each cycle in the graph. There is no known example of a pair of representations which gives a consistent tensor product graph when there is no R-matrix. There is no explanation of this observation either.

The Freudenthal magic square (from [Fre64]) is a 4×4 square of semi-simple Lie algebras. The rows and columns are labelled by the four real normed division algebras; the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. For an introduction to the octonions and the exceptional Lie algebras, see [Bae02]. The corresponding entry in the square is a Lie algebra constructed in a uniform way from the ordered pair of algebras. This is discussed in [Vin94, BS02, LM01]. In this paper we extend this square to a rectangle by adding three more columns.

Here is the extended magic square. In this square, each column has a uniform construction given an algebra. For the column labelled $m = -2/3$ the construction is to take the derivation algebra. For the column labelled $m = 0$ the construction is to take the triality algebra. Taking the columns labelled $m = 1, 2, 4, 8$ and the rows labelled $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ gives the original Freudenthal magic square. The column labelled $m = 6$ is work in progress but will be discussed below.

	$-2/3$	0	1	2	4	6	8
$\mathbb{R}$	0	0	A_1	A_2	C_3	$C_3.H_{14}$	F_4
$\mathbb{C}$	0	T_2	A_2	$2A_2$	A_5	$A_5.H_{20}$	E_6
$\mathbb{H}$	A_1	$3A_1$	C_3	A_5	D_6	$D_6.H_{32}$	E_7
$\mathbb{S}$	$A_1.H_4$	$(3A_1).H_8$	$C_3.H_{14}$	$A_5.H_{20}$	$D_6.H_{32}$	$D_6.H_{32}.H_{44}$	$E_7.H_{56}$
$\mathbb{O}$	G_2	D_4	F_4	E_6	E_7	$E_7.H_{56}$	E_8

The convention here is that a Lie algebra $G.H_{2n}$ means that the Lie algebra of type G has a representation V of dimension $2n$ which admits an invariant symplectic form ω. Then G acts on the Heisenberg algebra of (V, ω) and $G.H_{2n}$ denotes the semi-direct product. These algebras are not reductive and the Heisenberg algebra is the radical.

These non-reductive Lie algebras are explained by the sextonion algebra. This is a six-dimensional alternative algebra intermediate between the split quaternions and the split octonions. It does not admit a compact form. It has a degenerate norm; the kernel of the norm is the radical of the algebra. This ideal is two dimensional and the quotient is the quaternion algebra. In this paper we do not discuss the sextonion algebra in any detail. However, we have mentioned this because it does predict new trigonometric solutions of the Yang–Baxter equation. The vector spaces for these two solutions have dimensions 21 and 44. This also predicts two new rational solutions of the Yang–Baxter equation. The vector spaces for these solutions have dimensions 100 and 190. These examples involve non-reductive groups. However, there is a precedent for introducing non-reductive groups namely the odd symplectic groups of [GZ84]. The rational solutions of the Yang–Baxter equation can be constructed in all these cases from the representation theory of these non-reductive Lie algebras. However the trigonometric solutions are conjectural as the quantum groups associated with these non-reductive Lie algebras have not been constructed.

The main result of this paper is to show that each row of this square (except the first) has a uniform tensor product graph. This means that we can construct a tensor product graph with edges labelled by linear functions in m. Then evaluating these linear functions by taking m to be the dimension of the algebra gives the tensor product graphs in each row. This has been extended in [Mac02] who shows that the K-matrices which satisfy the reflection equation and are associated with symmetric spaces also have tensor product graphs which are uniform in the same way.

Next we discuss conventions which will be in force throughout the paper. All of these conventions have precedents in the literature.

For the two rows corresponding to $\mathbb{C}$ and $\mathbb{H}$ we will give linear functions of m for the dimension of the representation. For some values of m this may give a negative integer. In these cases the representation is to be understood as a super vector space with zero even part. In practice, the effect of this is to make some dimensions negative and also to interchange the roles of the symmetric and alternating powers of the representation. The best known precedent for this is that the symplectic groups $Sp(2n)$ can be regarded as $SO(-2n)$ by considering them as the automorphism groups of an odd vector space with an invariant symmetric inner product. This principle is also discussed in [Cvi81].

Another convention which was introduced in [Del96] is that although we will use the notation of highest weight vectors for representations, the representations we will be working with are in fact representations of the automorphism group of the Lie algebra of the Dynkin diagram. This automorphism group is the semi-direct product of the semi-simple group of adjoint type associated with the diagram with the finite group of diagram automorphisms. The exception to this is that in the row associated with $\mathbb{C}$ there is a diagram automorphism which interchanges the highest weight of the distinguished representation and its dual. In this case we work with a subgroup of index two in the automorphism group so that we still have an involution.

Finally we explain some notation.

Notation 1.1. *If V and W are representations with highest weights λ and μ then we will write VW for the representation with highest weight $\lambda + \mu$ and V^p for the representation with highest weight $p\lambda$.*

We write highest weights in terms of the basis of fundamental weights and the ordering of the nodes of the Dynkin diagram is the one used by Bourbaki.

2. Tensor product graphs

In this section we discuss the tensor product graphs of the rows of the magic rectangle. We have chosen to treat these in the following sequence: $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\mathbb{R}$. The reason for choosing this unusual sequence is that the first two cases, $\mathbb{C}$ and $\mathbb{H}$, have uniform tensor product graphs with edges labelled by linear functions of the dimension of the corresponding division algebra. For the other two cases this does not work. However, for the $\mathbb{O}$ case there is a uniform R-matrix for $\mathfrak{g} \oplus 1$, but this is not constructed from a tensor product graph. The remaining case, $\mathbb{R}$, is possibly the most interesting as there is no known uniform R-matrix.

2.1. Complex numbers

The Lie algebras are

m	-2	$-2/3$	0	1	2	4	6	8
G	A_2	0	T_2	A_2	$A_2 \oplus A_2$	A_5	$A_5.H_{20}$	E_6

There is a distinguished representation V of dimension $(3m + 3)$.

The common structure to these representations is

$$\Lambda^2(V) = V_2 \qquad S^2(V) = V^* \oplus V^2$$

where the representation V^2 is given by notation 1.1. The highest weight vectors of these representations are given in the following table:

	A_2	A_2	$2A_2$	A_5	E_6
V	$-[1,0]$	$[2,0]$	$[0,1\|1,0]$	$[0,1,0,0,0]$	$[1,0,0,0,0,0]$
$\mathfrak{g}$	$[1,1]$	$[1,1]$	$[1,1\|0,0]$	$[1,0,0,0,1]$	$[0,1,0,0,0,0]$
V_2	$-[2,0]$	$[2,1]$	$[1,0\|2,0]$	$[1,0,1,0,0]$	$[0,0,1,0,0,0]$

The Lie algebra $A_5.H_{20}$ is graded and the representations we are considering can be taken to be graded A_5 representations. These are given in the following table:

	0	1	2
$\mathfrak{g}$	$\lambda_1+\lambda_5$	λ_3	1
V	λ_2	λ_5	
V^*	λ_4	λ_1	
V^2	$2\lambda_2$	$(\lambda_2+\lambda_5)$	λ_5
V_2	$(\lambda_1+\lambda_3)$	$(\lambda_2+\lambda_5)\oplus\lambda_1$	λ_4

The quantum dimensions are given by

$$\dim_q(V) = \frac{[3m/2][m+1]}{[m/2]}$$

$$\dim_q(V^2) = \frac{[m+2][m+1][3m/2+1][3m/2]}{[2][m/2+1][m/2]}$$

$$\dim_q(V_2) = \frac{[m+1][m-2][3m/2][3m/2+1]}{[2][m/2][m/2-1]}$$

$$\dim_q(\mathfrak{g}) = \frac{[m-2][m][m+1][3m/2+1]}{[m/2-1][m/2][m/2+2]}.$$

In particular, $\dim(\mathfrak{g})$ and $\dim(V)$ are integers if and only if $3m+12$ divides 360. This gives a finite list of possible values of m.

The values of the Casimir are

V^p	$4p^2/3+2pm$
V_2V^p	$4p^2/3+2p(3m+5)/3+4(3m+1)/3$
V^*V^p	$4p^2/3+2p(3m+2)/3+2(3m+2)/3$
$\mathfrak{g}V^p$	$4p^2/3+2p(m+1)+3m$

The tensor product graph for $V\otimes V^p$ is

$$V^{p+1} \xrightarrow{2p+2} V_2V^{p-1} \xrightarrow{2m+2p-2} V^*V^{p-1}$$

where the representations V^p are given by notation 1.1.

The tensor product graph for $V^*\otimes V^p$ is

$$V^*V^p \xrightarrow{m+2p+2} \mathfrak{g}V^{p-1} \xrightarrow{3m+2p-2} V^{p-1}.$$

These tensor product graphs for $m=8$ are given in [DGZ94]. The R-matrix for E_6 is given in [CK91, ZGB91, Ser91].

2.2. Quaternions

The standard Lie algebras are

m	$-2/3$	0	1	2	4	6	8
G	A_1	$3A_1$	C_3	A_5	D_6	$D_6.H_{32}$	E_7

This list can be extended as follows:

m	-3	$-8/3$	$-5/2$	-2	$-4/3$	-1
G	D_5	B_3	G_2	$2A_1$	0	$D(2,1,\alpha)$

where $D(2,1,\alpha)$ is an exceptional simple super Lie algebra. This series was first studied in [Cvi81].

There is a distinguished representation V of dimension $(6m+8)$. The structure that these representations have in common is that

$$\Lambda^2(V) = 1 \oplus V_2 \qquad S^2(V) = \mathfrak{g} \oplus V^2$$

where V^2 is given by notation 1.1. The highest weight vectors of these representations are given in the following table:

	A_1	$3A_1$	C_3	A_5	D_6	E_7
V	[3]	[1, 1, 1]	[0, 0, 1]	[0, 0, 1, 0, 0]	[0, 0, 0, 0, 0, 1]	[0, 0, 0, 0, 0, 0, 1]
V_2	[4]	[2, 2, 0]	[0, 2, 0]	[0, 1, 0, 1, 0]	[0, 0, 0, 1, 0, 0]	[0, 0, 0, 0, 0, 1, 0]

The Lie algebra $D_6.H_{32}$ is graded and the representations we are considering can be taken to be graded D_6 representations. These are given in the following table:

	0	1	2
$\mathfrak{g}$	λ_2	λ_5	1
V	λ_6	λ_1	
V^2	$2\lambda_6$	$(\lambda_1+\lambda_6)$	$2\lambda_1$
V_2	λ_4	$(\lambda_1+\lambda_6)$	λ_2

The representations V in the extension are given by

	D_5	B_3	G_2	$2A_1$
V	$-[0,0,0,0,1]$	$-[0,0,1]$	$-[1,0]$	$-[1]\otimes[1]$

The quantum dimensions of these representations are given by

$$\dim_q(\mathfrak{g}) = \frac{[2m+3][3m/2+2][3m/2]}{[m/2][m/2+2]}$$

$$\dim_q(V) = \frac{[m+2][3m/2+2][2m+2]}{[m/2+1][m+1]}$$

$$\dim_q(V^2) = \frac{[m+3][3m/2+2][3m/2+3][2m+2][2m+3]}{[2][m/2+1][m/2+2][m+1]}$$

$$\dim_q(V_2) = \frac{[2m+3][2m+2][3m/2][3m/2+3]}{[2][m/2][m/2+1]}.$$

In particular, $\dim(V)$ and $\dim(\mathfrak{g})$ are both integers if and only if $(6m+24)$ divides 720. This gives a finite list of possibilities. If the projection $\Lambda^2(V) \to 1$ is a symplectic form then $\dim(V)$ must be an even integer and this then restricts attention to those values of m such that $(3m+12)$ divides 360.

The tensor product graph for $V \otimes V^p$ is the following:

$$V^{p-1} \xrightarrow{2m+p+1} \mathfrak{g}V^{p-1} \xrightarrow{m+p+1} V_2V^{p-1} \xrightarrow{p+1} V^{p+1}$$

where the representations V^p are given by notation 1.1.

The R-matrix for E_7 is given in [KKM91, JM95, CK91, ZGB91, Ser91]. The R-matrix for G_2 is given in [Kun90, Ma90].

The values of the Casimir are

V^p	$3p^2/4 + p(3m+3)/2$
$\mathfrak{g}V^p$	$3p^2/4 + p(3m+5)/2 + 2m + 2$
V_2V^p	$3p^2/4 + p(3m+7) + 3m + 4$

The representations in the extension have the common property that

$$\Lambda^2(V) = \mathfrak{g} \oplus V_2 \qquad S^2(V) = 1 \oplus V^2.$$

The interpretation that justifies including them in this series is to regard V as a super vector space with zero even part. This has the effect of making the dimension—$\dim(V)$ and interchanges the symmetric and exterior powers. This gives the same tensor product graph (except for G_2 and $p > 1$) but not the labels on the edges.

2.3. Octonions

The exceptional series of Lie algebras are

m	$-5/2$	$-3/2$	$-4/3$	-1	$-2/3$	0	1	2	4	6	8
G	1	$OSP(2,1)$	A_1	A_2	G_2	D_4	F_4	E_6	E_7	$E_7.H_{56}$	E_8

where $OSP(2,1)$ is a simple super Lie algebra of dimension $(3,2)$. This is a simple super Lie algebra whose finite-dimensional representations are all completely reducible.

The Lie algebras in this list are the exceptional series of Lie algebras discussed in [DdM96, CdM96, Del96]. However, this series has no distinguished representation, other than the adjoint representation. In [Vog99] evidence is given for a family of Lie algebras depending on three homogeneous parameters. This family includes all the simple Lie algebras and includes the series of exceptional series as a straight line. In this section we will discuss the adjoint representation from the point of view of R-matrices. Our conclusion is that the results of [CP91] fit in nicely with this point of view.

The structure common to the adjoint representation is that

$$\Lambda^2(\mathfrak{g}) = \mathfrak{g} \oplus X \qquad S^2(\mathfrak{g}) = 1 \oplus Y(\alpha) \oplus Y(\beta) \oplus Y(\gamma).$$

Here the projection $\Lambda^2(\mathfrak{g}) \to \mathfrak{g}$ is the Lie bracket and the projection $S^2(\mathfrak{g}) \to 1$ is the Killing form. This structure is given in [Mey84].

The weights of these representations for the classical groups are given by the following table. For the corresponding weights for the exceptional groups see [CdM96].

G	$\mathfrak{g}$	X	$Y(\alpha)$	$Y(\beta)$	$Y(\gamma)$
$SL(n)$	$\lambda_1+\lambda_n$	$\lambda_2+2\lambda_n$	$2\lambda_1+2\lambda_2$	$\lambda_1+\lambda_n$	$\lambda_2+\lambda_{n-1}$
$SO(n)$	λ_2	$\lambda_1+\lambda_3$	$2\lambda_2$	λ_4	$2\lambda_2$
$Sp(2n)$	$2\lambda_1$	$2\lambda_1+\lambda_2$	$2\lambda_2$	$4\lambda_1$	λ_2

In this situation we do not have an obvious parameter, m. Instead, we take the values of the Casimir on $Y(\alpha)$, $Y(\beta)$ and $Y(\gamma)$ as homogeneous parameters (α, β, γ). In this paper, we assume that these are non-zero.

Note that there is an action of the permutation group S_3. This group acts by applying a permutation to the coordinates (α, β, γ) and to the representations $(Y(\alpha), Y(\beta), Y(\gamma))$. The preprint [Vog99] takes the quotient by this action.

The quantum dimensions are given by

$$\dim_q(\mathfrak{g}) = \frac{[\alpha+2\beta+2\gamma][2\alpha+\beta+2\gamma][2\alpha+2\beta+\gamma]}{[\alpha][\beta][\gamma]}$$

(1)

$$\dim_q(X) = \frac{[\alpha+\beta+2\gamma][\alpha+2\beta+\gamma][2\alpha+\beta+\gamma]}{[2\alpha][2\beta][2\gamma]} \times \frac{[\alpha+2\beta+2\gamma][2\alpha+\beta+2\gamma][2\alpha+2\beta+\gamma]}{[\alpha][\beta][\gamma]} \frac{[2\alpha+2\beta][2\alpha+2\gamma][2\beta+2\gamma]}{[\alpha+\beta][\alpha+\gamma][\beta+\gamma]}$$

(2)

$$\dim_q(Y(\alpha)) = -\frac{[2\alpha+2\beta+2\gamma][2\alpha+2\beta+\gamma][2\alpha+\beta+2\gamma]}{[2\alpha][\alpha][\beta][\gamma]} \times \frac{[\alpha+\beta+2\gamma][\alpha+2\beta+\gamma][\alpha-2\beta-2\gamma]}{[\alpha-\beta][\alpha-\gamma]}.$$

Although the exceptional series does not have a preferred representation (other than the adjoint representation) it is distinguished by the fact that

$$S^2(\mathfrak{g}) = 1 \oplus Y \oplus \mathfrak{g}^2$$

or by the property that there is no quartic Casimir.

This plane has a number of lines:

	α	β	γ	
$SL(n)$	-2	2	n	$\alpha+\beta=0$
$SO(n)$	-2	4	$n-2$	$2\alpha+\beta=0$
$Sp(2n)$	-2	4	$-2n-2$	$2\alpha+\beta=0$
$\mathbb{O}$	-2	$m+4$	$2m+4$	$2\alpha+2\beta-\gamma=0$
$\mathbb{H}$	-2	m	$m+4$	$2\alpha-\beta+\gamma=0$

Note that here $Sp(2n)$ can be regarded as $SO(-2n)$. There is one difference which is that $\mathfrak{g}^2$ is $Y(\alpha)$ for $SO(n)$ and is $Y(\beta)$ for $Sp(n)$.

The Lie algebra $SL(2)$ is a degenerate case and satisfies $\Lambda^2(\mathfrak{g}) = \mathfrak{g}$ and $S^2(\mathfrak{g}) = 1 \oplus \mathfrak{g}^2$. This does not determine the values of the three parameters and in fact all points on the line $\alpha + \beta + 2\gamma = 0$ give the Lie algebra $SL(2)$.

Another line, which we do not discuss in this paper, is the line $\alpha + \beta + \gamma = 0$ which gives the series of super Lie algebras D(2, 1, α).

The odd symplectic Lie algebra $Sp(2n+1) = Sp(2n).H_{2n}$ is graded and the representations we are considering can be taken to be graded $Sp(2n)$ representations. These are given in the following table:

	0	1	2	3	4
$\mathfrak{g}$	$2\lambda_1$	λ_1	1		
X	$(2\lambda_1 + \lambda_2)$	$(\lambda_1 + \lambda_2) \oplus 3\lambda_1$	$\lambda_2 \oplus 2\lambda_1$	λ_1	
$Y(\alpha)$	$2\lambda_2$	$(\lambda_1 + \lambda_2)$	$2\lambda_1$	$\lambda - 1$	1
$Y(\beta)$	$4\lambda_1$	$3\lambda_1$			
$Y(\gamma)$	λ_2	λ_1			

The dimensions of these representations of $Sp(n)$ are

$$\dim(\mathfrak{g}) = n(n+1)/2$$
$$\dim(X) = (n-2)n(n+1)(n+3)/8$$
$$\dim(Y(\alpha)) = (n-2)(n-1)n(n+3)/12$$
$$\dim(Y(\beta)) = n(n+1)(n+2)(n+3)/24$$
$$\dim(Y(\gamma)) = (n-2)(n+1)/2.$$

The Lie algebra $E_7.H_{56}$ is graded and the representations we are considering can be taken to be graded E_7 representations. These are given in the following table:

	0	1	2	3	4	dim
$\mathfrak{g}$	λ_1	λ_7	1			190
X	λ_3	$(\lambda_1 + \lambda_7) \oplus \lambda_2$	$\lambda_1 \oplus \lambda_6$	λ_7		17 765
$\mathfrak{g}^2$	$2\lambda_1$	$(\lambda_1 + \lambda_7)$	$\lambda_1 \oplus 2\lambda_7$	λ_7	1	15 504
Y	λ_6	$\lambda_2 \oplus \lambda_7$	λ_1			2 640

The Lie algebra $D_6.H_{32}$ is graded and the representations we are considering can be taken to be graded D_6 representations. These are given in the following table:

	0	1	2	3	4	dim
$\mathfrak{g}$	λ_2	λ_5	1			99
X	$(\lambda_1 + \lambda_3)$	$(\lambda_2 + \lambda_5) \oplus (\lambda_1 + \lambda_6)$	$\lambda_2 \oplus \lambda_4$	λ_5		4752
$Y(\alpha)$	$2\lambda_2$	$(\lambda_1 + \lambda_5)$	$\lambda_2 \oplus 2\lambda_5$	λ_5	1	3927
$Y(\beta)$	λ_4	$\lambda_5 \oplus (\lambda_1 + \lambda_6)$	λ_2			945
$Y(\gamma)$	$2\lambda_1$					77

In addition to these lines there is an isolated point with parameters $(-1, 4, 7)$ (and its permutations). This Lie algebra has dimension 156. The candidate for this Lie algebra is a centralizer of a unipotent element of E_8 (see [Car93]). This is a graded Lie algebra where the dimensions of the non-zero components are given by

0	1	2
78	64 + 13	1

The Levi subalgebra is the degree zero subalgebra and is B_6.

The tensor product graph is not consistent and so this does not give an R-matrix. However, in [CP91] it is shown that there is a rational R-matrix for $\mathfrak{g} \oplus 1$. This R-matrix is given by

$$R(x) = P_X \oplus \left(\frac{\alpha + x}{\alpha - x}\right) P_{Y(\alpha)} \oplus \left(\frac{\beta + x}{\beta - x}\right) P_{Y(\beta)} \oplus \left(\frac{\gamma + x}{\gamma - x}\right) P_{Y(\gamma)} \oplus P_{\mathfrak{g}} \oplus P_1.$$

Introduce the notation

$$a = \alpha + \beta + \gamma \qquad b = \alpha\beta\gamma \qquad h(x) = (x - \alpha)(x - \beta)(x - \gamma).$$

Then the matrices $P_{\mathfrak{g}}$ and P_1 are given by

$$P_{\mathfrak{g}} = \frac{1}{h(x)} \begin{pmatrix} -h(x) + 2b & -2x & 0 \\ 2abx & -h(-x) + 2b & 0 \\ 0 & 0 & h(x) \end{pmatrix}$$

and the matrix P_1 is given by

$$P_1 = \frac{1}{h(x)} \begin{pmatrix} -h(x) + 2b & bx/(x - a) \\ 4ax(x + a) & (h(-x) + 2b)(x + a)/(x - a) \end{pmatrix}.$$

Although the R-matrices have not been worked out, it is also shown in [CP91] that for the exceptional series there are the following uniform minimal affinisations:

$$V_a(Y) = Y \oplus \mathfrak{g} \oplus 1$$
$$V_a(X) = X \oplus Y^* \oplus 2\mathfrak{g} \oplus 1.$$

These are obtained by a fusion argument. This argument generalizes to give the uniform affinisations

$$V_a(Y(\alpha)) = Y(\alpha) \oplus \mathfrak{g} \oplus 1$$
$$V_a(X) = X \oplus Y(\alpha) \oplus 2\mathfrak{g} \oplus 1$$

and similarly with α replaced by β and γ.

2.4. Real numbers

The first row of the Freudenthal magic square gives the following Lie algebras:

m	$-2/3$	0	1	2	4	6	8
G	0	0	A_1	A_2	C_3	$C_3.H_{14}$	F_4

These Lie algebras can be constructed uniformly in terms of a division algebra $\mathbb{A}$ by taking the derivation algebra of the exceptional Jordan algebra $H_3(\mathbb{A})$. There is a distinguished representation V of dimension $(3m + 2)$. This representation can be constructed as the space of trace-free, anti-Hermitian 3×3 matrices with entries in $\mathbb{A}$.

The structure that these representations have in common is that

$$\Lambda^2(V) = \mathfrak{g} \oplus V_2 \qquad S^2(V) = \mathbb{C} \oplus V \oplus V^2$$

where the representation V^2 is given by notation 1.1. In particular, these representations have invariant symmetric bilinear and trilinear forms. They also have an invariant symmetric form of degree four.

The highest weights of these representations are given in the following table:

	A_1	A_2	C_3	F_4
V	[4]	[1, 1]	[0, 1, 0]	[0, 0, 0, 1]
$\mathfrak{g}$	[2]	[1, 1]	[2, 0, 0]	[1, 0, 0, 0]
V_2	[6]	[3, 0]	[1, 0, 1]	[0, 0, 1, 0]

The Lie algebra $C_3.H_{14}$ is graded and the representations we are considering can be taken to be graded C_3 representations. These are given in the following table:

	0	1	2
$\mathfrak{g}$	$2\lambda_1$	λ_3	1
V	λ_2	λ_1	
V^2	$2\lambda_2$	$(\lambda_1+\lambda_2)$	λ_3
V_2	$\lambda_1+\lambda_3$	$\lambda_1 \oplus (\lambda_1+\lambda_2)$	λ_2

The quantum dimensions are given by

$$\dim_q(V) = \frac{[m/4+1]}{[m/2+2]}\frac{[5m/2-2]}{[5m/4-1]}\frac{[m]}{[m/2]}[3m/2+1]$$

$$\dim_q(\mathfrak{g}) = \frac{[m/4+1]}{[m/2+2]}\frac{[m/4+2]}{[m/2+4]}\frac{[m/4-1]}{[m/2-2]}\frac{[5m/2]}{[5m/4]} \times \frac{[3m/2-2]}{[3m/4-1]}\frac{[3m/2-6]}{[3m/4-3]}\frac{[3m/2]}{[m/2]}\frac{[m][3m/2+1]}{[m/2+2]}$$

$$\dim_q(V^2) = \frac{[m/4+2]}{[m/2+4]}\frac{[m/4+1]}{[m/2+2]}\frac{[5m/2-2]}{[5m/4-1]}\frac{[5m/2]}{[5m/4]} \times \frac{[3m/2]}{[m/2]}\frac{[3m/2+3]}{[m/2+1]}\frac{[m][m+1]}{[2]}$$

$$\dim_q(V_2) = \frac{[5m/2-2]}{[5m/4-1]}\frac{[5m/2]}{[5m/4]}\frac{[m/4]}{[m/2]}\frac{[m/4+1]}{[m/2+2]} \times \frac{[m-2]}{[m/2-1]}\frac{[3m/2-2]}{[3m/4-1]}\frac{[m+1][3m/2+1][3m/4+1]}{[2][m/2+2]}.$$

In particular, $\dim(V)$ and $\dim(\mathfrak{g})$ are both integers if and only if $3m+12$ divides 360. This gives a finite list of possible values of m. Some of these can be eliminated using other dimension formulae.

The values of the Casimir are

$$\begin{array}{ccccc} 1 & \mathfrak{g} & V^2 & V_2 & V \\ 0 & 5m-4 & 6m+4 & 6m & 3m \end{array}.$$

The tensor product graphs are given by

$$F_4 : 1 \xrightarrow{36} \mathfrak{g} \xrightarrow{-12} V^2 \xrightarrow{24} V_2 \xrightarrow{4} V$$

This R-matrix is given in [KKM91, Ma91, Ser91, DGZ94].

$$\begin{tikzcd} & & V \arrow[dr, "-4"] & \\ C_3 : 1 \arrow[r, "16"] & \mathfrak{g} \arrow[ur, "12"] \arrow[dr, "-4"'] & & V_2. \\ & & V^2 \arrow[ur, "12"'] & \end{tikzcd}$$

This R-matrix is given in [Mac92].

$$A_1 : 1 \xrightarrow{1} \mathfrak{g} \xrightarrow{9} V \xrightarrow{-4} V_2 \xrightarrow{-3} V^2.$$

This leads us to consider the tensor product graph

$$\begin{tikzcd} & & V \arrow[dr, "3m"] & \\ 1 \arrow[r, "5m-4"] & \mathfrak{g} \arrow[ur, "-2m+4"] \arrow[dr, "m+8"'] & & V_2. \\ & & V^2 \arrow[ur, "-4"'] & \end{tikzcd}$$

However, this tensor product graph is consistent if and only if $m = 4$ or $m = -4/3$. The conclusion is that although we have a tensor product graph this is not consistent and so does not give an R-matrix. It may be that these representations have affinisations which give a uniform R-matrix.

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